JOURNAL OF APPROXIMATION THEORY **85**, 343–347 (1996)
ARTICLE NO. 0047

Another Note on Polynomial vs Rational Approximation

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Communicated by Peter B. Borwein

Received October 17, 1994; accepted June 6, 1995

Let E be a subspace of $C(X)$ and let $R(E) = g/h: g,\ h \in E; h > 0\}$. We make a simple, yet intriguing observation: if zero is a best approximation to f from E, then zero is a best approximation to f from $R(E)$.

We also prove that if $\{E_n\}$ is dense in $C(X)$ then for almost all f (in the sense of category)

$$\limsup d(f, R(E_n))/d(f, E_n) = 1.$$

That extends the results of P. Borwein and S. Zhou who proved it for the case when E_n is the space of algebraic or trigonometric polynomials of degree n.

1. INTRODUCTION

Consider an arbitrary function $f \in C_{[-1,1]}$. Let $\mathscr{P}_n$ stand for the space of polynomials of degree n and let $\mathscr{R}_{n,n}$ stand for rational functions

$$\left\{\frac{g}{h}: g,\ h \in \mathscr{P}_n;\ h > 0\right\}.$$

Let p_n^* be the best approximation to f from $\mathscr{P}_n$. Then zero is the unique best approximation from $\mathscr{R}_{n,n}$ to $f - p_n^*$.

Here is a short proof:

The function $f - p_n^*$ equioscillates i.e. there are points $\xi_1, ..., \xi_{n+2} \in [-1, 1]$ such that $(f - p_n^*)(\xi_j) = \lambda(-1)^j \|f - p_n^*\|$ where $\lambda = \pm 1$ (say $\lambda = -1$). Now if $\|f - p_n^* - g/h\| \leqslant \|f - p_n^*\|$ then $g/h(\xi_j) \geqslant 0$ for j even and $g/h(\xi_j) \leqslant 0$ for j odd. Since h is strictly positive, the function $g \in \mathscr{P}_n$ should satisfy the same condition

$$g(\xi_j) \geqslant 0 \quad \text{for } j \text{ even} \qquad \text{and} \qquad g(\xi_j) \leqslant 0 \quad \text{for } j \text{ odd}.$$

0021-9045/96 $18.00

That forces g to have $n+1$ zeros and hence $g=0$. Examining this proof it is easy to conclude that it has nothing to do with the nature of h, as long as it is strictly positive. The only property of g that we used is that g and p^* belong to the same Chebyshev subspace of $C_{[-1,1]}$.

It turns out that this statement (aside from uniqueness) holds true for rational functions where the numerator and denominator come from arbitrary subspaces of $C(X)$. This is the content of Theorem 2.1.

We then use this theorem to prove that for most of the functions (in the sense of category) in $C(X)$ the rate of best approximation and the rate of best rational approximation is the same. This is known for specific subspaces (cf [1], [3]). We prove it for arbitrary subspaces of $C(X)$.

2. The Best Rational Approximation

Let X be a compact Hausdorff space, let $C(X)$ be the space of real-valued continuous functions on X. If G and H are subspaces of $C(X)$ we use

$$R(G, H) := \{ g/h : g \in G,\ h \in H,\ h(x) > 0 \text{ for all } x \in X \}.$$

To avoid trivialities we will always assume that H contains a strictly positive function.

We will identifying the dual space $(C(X))^*$ with the space of regular Borel measures on X: $\mathscr{M}(X)$, and the same letter may mean a measure or a functional.

Finally if A is a subset of $C(X)$, and $f \in C(X)$

$$d(f, A) := \inf \|f - a\| : a \in A\}.$$

THEOREM 2.1. *Let $f \in C(X)$ and $G \subset C(X)$ be a subspace such that there exists $g \in G$ with $\|f - g^*\| = d(f, G)$. Then for every subspace $H \subset C(X)$*

$$d(f - g^*, R(G, H)) = \|f - g^*\|.$$

Hence zero is a best approximation to $f - g^*$ from $R(G, H)$. Moreover if $X = [a, b]$ and G is Chebyshev then zero is the unique best approximation from $R(G, H)$ to $f - g^*$.

Proof. Since g^* is the best approximation from G to f, hence there exists a functional $\mu \in \mathscr{M}(X)$ such that

$$\mu \perp G \text{ i.e. } \mu(g) = 0 \qquad \text{for all} \quad g \in G. \tag{1}$$

$$\|\mu\| = 1; \quad \mu(f - g^*) = \|f - g^*\| \ \ \|\mu\|. \tag{2}$$

We adopt the logic of [2] for this particular case.

Let $a>0$ be such that $d(f-g^*, R(G, H))<a$. Then there exists $g\in G$, $h\in H$, $h>0$ such that for $\tilde{f}:=g/h$ we have $\|(f-p^*)-\tilde{f}\|<a$.

On the other hand,

$$0\neq \|f-p^*\|(|\mu|\,(h))=((f+p^*)\mu)(h)=\int (f-p^*)\,h\,d\mu$$

$$=\int ((f-p^*)-\tilde{f})\,h\,d\mu+\int \tilde{f} h\,d\mu=\int ((f-p^*)-\tilde{f})\,h\,d\mu+\int g d\mu$$

$$(\text{since } \tilde{f}h=g)=\int ((f-p^*)-\tilde{f})\,h\,d\mu$$

$$(\text{since } \mu\perp G)\leqslant \|(f-p^*)-\tilde{f}\| \int hd\,|\mu|\ (\text{since } h \text{ is positive})<a\,|\mu|(h).$$

Thus $a>\|f-p^*\|=d(f, G)$. The "moreover" part of the Theorem was already proved in the Introduction. ▮

Remark. Theorem 2.1 is a very simple observation. Yet even in the simple case of $C(X)=C_{[-1,1]}$; $R(G, H)=\mathscr{R}_{n,n}$ it is somewhat surprising. First of all it provides a large class of functions for which the best rational approximation is easily computed.

Second, it shows how easy it is to spoil a function for rational approximation.

For instance $d(|x|, \mathscr{R}_{n,n})\sim e^{\alpha\sqrt{n}}$. Add to $|x|$ a polynomial of degree n (namely $-p_n^*$) and the rate of approximation drops, and drops significantly to $1/n$, since $d((|x|-p_n^*), \mathscr{R}_{n,n})=\|\,|x|-p_n^*\|\sim 1/n$.

3. Rates of Approximation

We now use the Theorem 2.1 to extend a result of P. Borwein and S. Zhou (cf. [1], Theorem 1) from $\mathscr{R}_{n,n}$ to $R(G_n, H_n)$ for arbitrary G_n, $H_n\subset C(X)$.

Theorem 3.1. *Let X be an infinite compact Hausdorff space. Let $G_n\subset C(X)$ be a sequence of finite-dimensional subspaces such that*

$$d(f, G_n)\to 0 \qquad \textit{for all} \quad f\in C(X).$$

Then for all finite-dimesnional subspaces $H_n\subset C(X)$, the set

$$A:=\{f\in C(X)\text{: lim sup}[d(f, R(G_n, H_n))/d(f, G_n)]\geqslant 1\}$$

is the set of second category in $C(X)$.

Proof. We proceed as in [1]. Let $\tilde{G}_n := C(X) \backslash G_n$. Then $\tilde{G}_n$ is open and dense in $C(X)$. We consider sets

$A_n = \{ f \in C(X)$: there exists $m(n) > n$ with

$$d(f, R(G_m, H_m))/d(f, G_m) > 1 - \frac{1}{n};\ d(f, G_m) \neq 0\}.$$

Then $A = (\bigcap_{n=1}^{\infty} A_n) \cap (\bigcap_{n=1}^{\infty} \tilde{G}_n)$. It remains to prove that each set A_n is open and dense in $C(X)$. The proof that A_n is open is exactly the same as in ([1], Theorem 1) and we refer to it for technical details. The idea, however is very simple. For a fixed $f \in A_n$ choose ε and δ so small that for all $\tilde{f}$ with $\|f - \tilde{f}\| < \delta$ we have

$$\tilde{f} \notin G_n; \quad |d(\tilde{f}, R(G_n, H_n)) - d(f, R(G_n, H_n))| < \varepsilon;$$
$$|d(f, G_n) - d(\tilde{f}, G_n)| < \varepsilon.$$

Since ε is "very small" the ratio $d(\tilde{f}, R(G_n H_n))/d(\tilde{f}, G_n)$ is still greater than $1 - 1/n$.

We now turn to the density of A_n. Let $f \in C(X)$. For arbitrary $\varepsilon > 0$ pick $\eta = \varepsilon/2$ and let $g_m \in G_m$ be such that $\|f - g_m\| < \eta$; $m > n$. Let

$$E_m := \operatorname{span}\{ g_m \cdot h_m + g'_m : g_m, g'_m \in G_m, h_m \in H_m\}.$$

Since G_m and H_m are of finite dimension, so is E_m. Let F be an arbitrary function in $C(X) \backslash E_m$. Since E_m is finite-dimensional, there exists e_m^* which is a best approximation to F from E_m. Denote

$$F := (F - e_m^*)/\|F - e_m^*\|.$$

We now consider the function

$$\varphi(x) = g_m(x) + \eta F^*(x).$$

Observe that $\|f - \varphi\| < \eta + \eta = \varepsilon$. It remains to show that $\varphi \in A_n$. Indeed, since $G_m \subset E_m$ we have

$$\eta \geqslant d(\varphi, G_m) = d(\eta F^*, G_m) \geqslant d(\eta F^*, E_m) = \eta.$$

Therefore

$$d(\varphi, G_m) = \eta. \tag{3}$$

Now let e_m/h_m be an arbitrary element in $R(E_m, H_m)$. Then

$$\varphi - \frac{e_m}{h_m} = \eta F^* + g_m - \frac{e_m}{h_m} = \eta F^* + \frac{g_m h_m - e_m}{h_m} = \eta F - \frac{e'_m}{h_m}$$

where $e'_m = e_m - g_m h_m$ is an arbitrary element in E_m. Thus

$$d(\varphi, R(E_m, H_m)) = d(\eta F^*, R(E_m, H_m)) = \eta.$$

The last equality follows from the Theorem 2.1.

Since $G_m \subset E_m$ we have

$$d(\varphi, R(G_m, H_m)) \geqslant d(\varphi, R(E_m, H_m)) = \eta;$$

which together with (3) implies

$$d(\varphi, R(G_m, H_m))/d(\varphi, G_m) \geqslant \eta/\eta = 1 > 1 - \frac{1}{n}$$

and hence $\varphi \in A_n$. ∎

Acknowledgment

I am grateful to Professor Zhou for pointing out the reference [1] and [3] to me. A conversation with him sparked off my interest in this subject.

References

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